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ON GLOBALLY SYMMETRIC FINSLER SPACES

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Abstract. The paper consider the symmetric of Finsler spaces. We give some conditions about globally symmetric Finsler spaces. Then we prove that these spaces can be written as a coset space of Lie group with an invariant Finsler metric. Finally, we prove that such a space must be Berwaldian.

1. Introduction

The study of Finsler spaces has important in physics and Biology ([5]), In particular there are several important books about such spaces (see [1], [8]). For example recently D. Bao, C. Robels, Z. Shen used the Randers metric in Finsler on Riemannian manifolds ([9] and [8], page 214). We must also point out there was only little study about symmetry of such spaces ([3], [12]). For example E. Cartan has been showed symmetry plays very important role in Riemannian geometry ([5] and [12], page 203).

Definition 1.1. *A Finsler space is locally symmetric if, for any $p \in M$, the geodesic reflection s_p is a local isometry of the Finsler metric.*

Definition 1.2. *A reversible Finsler space (M, F) is called globally Symmetric if for any $p \in M$ the exists an involutive isometry σ_x (that is, $\sigma_x^2 = I$ but $\sigma_x \neq I$) of such that x is an isolated fixed point of σ_x.*

Definition 1.3. *Let G be a Lie group and K is a closed subgroup of G. Then the coset space G/K is called symmetric if there exists an involutive automorphism σ*

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of G such that

$$G_\sigma^0 \subset K \subset G_\sigma,$$

where G_σ is the subgroup consisting of the fixed points of σ in G and G_σ^0 denotes the identity component of G_σ.

Theorem 1.4. *Let G/K be a symmetric coset space. Then any G-invariant reversible Finsler metric (if exists) F on G/K makes $(G/K, F)$ a globally symmetric Finsler space ([8], page 8).*

Theorem 1.5. *Let (M, F) be a globally Symmetric Finsler space. For $p \in M$, denote the involutive isometry of (M, F) at p by S_x. Then we have*

(a) For any $p \in M, (dS_x)_x = -I$. In particular, F must be reversible.

(b) (M, F) is forward and backward complete;

(c) (M, F) is homogeneous. This is, the group of isometries of $(M, F), I(M, F)$, acts transitively on M.

(d) Let $\widetilde{M}$ be the universal covering space of M and π be the projection mapping. Then $(\widetilde{M}, \pi^*(F))$ is a globally Symmetric Finsler space, where $\pi^*(F)$ is define by

$$\pi^*(F)(q) = F((d\pi)_{\widetilde{p}}(q)), \;\; q \in T_{\widetilde{p}}(\widetilde{M}),$$

(See [8] to prove).

Corollary 1.6. *Let (M, F) be a globally Symmetric Finsler space. Then for any $p \in M, s_p$ is a local geodesic Symmetry at p. The Symmetry s_p, is unique. (See prove of Theorem 1.2 and [1])*

2. A theorem on globally Symmetric Finsler spaces

Theorem 2.1. *Let (M, F) be a globally Symmetric Finsler space. Then exits a Riemannian Symmetric pair (G, K) such that M is diffeomorphic to G/K and F is invariant under G.*

Proof. The group $I(M, F)$ of isometries of (M, F) acts transitively on M (((C) of theorem 1.5). We proved that $I(M, F)$ is a Lie transformation group of M and for any $p \in M$ ([12] and [7], page 78), the isotropic subgroup $I_p(M, F)$ is a compact subgroup of $I(M, F)$ ([4]). Since M is connected ([7], [10]) and the subgroup K of G which p fixed is a compact subgroup of G. Furthermore, M is diffeomorphic to G/K under the mapping $gH \to g.p$, $g \in G$ ([7] Theorem 2.5, [10]).
As in the Riemannian case in page 209 of [7], we define a mapping s of G into G by $s(g) = s_p g s_p$, where s_p donote the (unique) involutive isometry of (M, f) with p as an isolated fixed point. Then it is easily seen that s is an involutive automorphism of G and the group K lies between the closed subgroup K_s of fixed points of s and

the identity component of K_s (See definition of the symmetric coset space, [11]). Furthermore, the group K contains no normal subgroup of G other than $\{e\}$. That is, (G,K) is symmetric pair. (G,K) is a Riemannian symmetric pair, because K is compact. □

The following useful will be results in the proof of our aim of this paper.

Proposition 2.2. *Let $(M,\bar{F})$ be a Finsler space, $p \in M$ and H_p be the holonomy group of $\bar{F}$ at p. If F_p is a H_p invariant Minkowski norm on $T_p(M)$, then F_p can be extended to a Finsler metric F on M by parallel translations of $\bar{F}$ such that F is affinely equivalent to $\bar{F}$ ([5], proposition 4.2.2)*

Proposition 2.3. *A Finsler metric F on a manifold M is a Berwald metric if and only if it is affinely equivalent to a Riemannian metric g. In this case, F and g have the same holonomy group at any point $p \in M$ (see proposition 4.3.3 of [5]).*

Now the main aim

Theorem 2.4. *Let (M,F) be a globally symmetric Finsler space. Then (M,F) is a Berwald space. Furthermore, the connection of F coincides with the Levi-civita connection of a Riemannian metric g such that (M,g) is a Riemannian globally symmetric space.*

Proof. We first prove F is Beraldian. By Theorem 2.1, there exists a Riemannian symmetric pair (G,K) such that M is diffeomorphic to G/K and F is invariant under G. Fix a G- invariant Riemannian metric g on G/K. Without losing generality, we can assume that (G,K) is effective (see [11] page 213). Since being a Berwald space is a local property, we can assume further that G/K is simple connected. Then we have a decomposition (page 244 of [11]):

$$G/K = E \times G_1/K_1 \times G_2/K_2 \times ... \times G_n/K_n,$$

where E is a Euclidean space, G_i/K_i are simply connected irreducible Riemannian globally symmetric spaces, $i = 1, 2, ..., n$. Now we determine the holonomy groups of g at the origin of G/K. According to the de Rham decomposition theorem ([2]), it is equal to the product of the holonomy groups of E and G_i/K_i at the origin. Now E has trivial holonomy group. For G_i/K_i, by the holonomy theorem of Ambrose and Singer ([12], page 231, it shows, for any connection, how the curvature form generats the holonomy group), we know that the lie algebra η_i of the holonomy group H_i is spanned by the linear mappings of the form $\{\widetilde{\tau}^{-1}R_0(X,Y)\widetilde{\tau}\}$, where τ denotes any piecewise smooth curve starting from o, $\widetilde{\tau}$ denotes parallel displacements (with respect to the restricted Riemannian metric) a long $\widetilde{\tau}$, $\widetilde{\tau}^{-1}$ is the inverse of $\widetilde{\tau}$, R_0 is the curvature tensor of G_i/K_i of the restricted Riemannian metric and $X, Y \in$

$T_0(G_i/K_i)$. Since G_i/K_i is a globally symmetric space, the curvature tensor is invariant under parallel displacements (page 201 of [10],[11]). So

$$\eta_i = span\{R_0(X,Y)|X,Y \in T_0(G_i/K_i)\},$$

(see page 243 of [7], [11]).
On the other hand, Since G_i is a semisimple group. We know that the Lie algebra of $K_i^* = Ad(K_i) \simeq K$ is also equal to the span of $R_0(X,Y)$ ([11]). The groups H_i, K_i^* are connected (because G_i/K_i is simply connected) ([10] and [11]). Hence we have $H_i = K_i^*$. Consequently the holonomy group H_0 of G/K at the origin is

$$K_1^* \times K_2^* \times ... \times K_n^*$$

Now F defines a Minkowski norm F_0 on $T_0(G/K)$ which is invariant by H_0 ([2]). By proposition 2.2, we can construct a Finsler metric $\bar{F}$ on G/K by parallel translations of g. By proposition 2.3, $\bar{F}$ is Berwaldian. Now for any point $p_0 = aK \in G/K$, there exists a geodesic of the Riemannian manifold $(G/K, g)$, say $\gamma(t)$ such that $\gamma(0) = 0, \gamma(1) = p_0$. Suppose the initial vector of γ is X_0 and take $X \in p$ such that $d\pi(X) = X_0$. Then it is known that $\gamma(t) = \exp tX.p_0$ and $d\tau(\exp tX)$ is the parallel translate of $(G/K, g)$ along γ ([11] and [7], page 208). Since F is G- invariant, it is invariant under this parallel translate. This means that F and $\bar{F}$ concede at $T_{p_0}(G/K)$. Consequently they concide everywhere. Thus F is a Berwald metric.
For the next assertion, we use a result of Szabo' ([2], page 278) which asserts that for any Berwald metric on M there exists a Riemannian metric with the same connection. We have proved that (M, F) is a Berwald space. Therefore there exists a Riemannian metric g_1 on M with the same connection as F. In [11], we showed that the connection of a globally symmetric Berwald space is affine symmetric. So (M, F) is a Riemannian globally symmetric space ([7], [11]). □

From the proof of theorem 2.4, we have the following corollary.

Corollary 2.5. *Let $(G/K, F)$ be a globally symmetric Finsler space and $g = \ell + p$ be the corresponding decomposition of the Lie algebras. Let π be the natural mapping of G onto G/K. Then $(d\pi)_e$ maps p isomorphically onto the tangent space of G/K at $p_0 = eK$. If $X \in p$, then the geodesic emanating from p_0 with initial tangent vector $(d\pi)_e X$ is given by*

$$\gamma_{d\pi.X}(t) = \exp tX.p_0.$$

Furthermore, if $y \in T_{p_0}(G/K)$, then $(d\exp tX)_{p_0}(Y)$ is the parallel of Y along the geodesic (see [11], [7] proof of theorem 3.3).

Example 2.6. *Let G_1/K_1, G_2/K_2 be two symmetric coset spaces with K_1, K_2 compact (in this coset, they are Riemannian symmetric spaces) and g_1, g_2 be invariant*

Riemannian metric on G_1/K_1, G_2/K_2, *respectively. Let* $M = G_1/K_1 \times G_2/K_2$ *and* O_1, O_2 *be the origin of* $G_1/K_1, G_2/K_2$, *respectively and denote* $O = (O_1, O_2)$ *(the origin of* M*). Now for* $y = y_1 + y_2 \in T_O(M) = T_{O_1}(G_1/K_1) + T_{O_2}(G_2/K_2)$, *we define*

$$F(y) = \sqrt{g_1(y_1, y_2) + g_2(y_1, y_2) + \sqrt[s]{g_1(y_1, y_2)^s + g_2(y_1, y_2)^s}},$$

where s *is any integer* ≥ 2. *Then* $F(y)$ *is a Minkowski norm on* $T_O(M)$ *which is invariant under* $K_1 \times K_2$ *([4]). Hence it defines an* G*- invariant Finsler metric on* M *([6], Corollary 1.2, of page 8246). By theorem 2.1, (M,F) is a globally symmetric Finsler space. By theorem 2.4 and ([2], page 266)* F *is non-Riemannian.*

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